INTEGRATING THE HAMILTON-JACOBI EQUATION FOR A CERTAIN CLASS OF HAMILTONIANS

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L.G. GLIKMAN
(Alma-Ata)

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In [1] we integrated the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial r}\right)^2+F=0$$

where S is an action function and F is a given function of the two variables r and t. The total integral of this equation was determined with certain conditions imposed on the function F. We shall now extend the method of [1] to the case of several variables.

We shall attempt to find the solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial S}{\partial q_i}\right)^2+F(q_1, q_2, \ldots, q_n, t)=0 \tag{1}$$

in the form

$$S=S_0+S_1, \qquad S_0=\sum_{i=1}^{n}X_{0i}(x_i)\,T_i(t), \quad S_1=\sum_{i=1}^{n}X_{1i}(x_i)+T_{n+1}(t) \tag{2}$$

Let us introduce the new variables $x_1=q_1f,\ x_2=q_2f,\ldots,\ x_n=q_nf$; here $f=f(t)$ is any doubly differentiable function of t. Substituting (2) into (1), we obtain

$$\frac{1}{f^2}\frac{\partial S_0}{\partial t}+\frac{1}{f^2}\frac{\partial S_1}{\partial t}+\frac{1}{f^2}F+\sum_{i=1}^{n}\left[\frac{\partial S_1}{\partial x_i}\left(2\frac{\partial S_0}{\partial x_i}+\frac{x_i}{f^3}\frac{df}{dt}\right)+\right.$$
$$\left.+\frac{\partial S_0}{\partial x_i}\frac{x_i}{f^3}\frac{df}{dt}+\left(\frac{\partial S_0}{\partial x_i}\right)^2+\left(\frac{\partial S_1}{\partial x_i}\right)^2\right]=0 \tag{3}$$

Let us require that the coefficients of $\partial S_1/\partial x_i$ equal zero. We then have

$$S_0=-\frac{r^2}{4f^3}\frac{df}{dt} \qquad \left(r^2=\sum_{i=1}^{n}x_i^2\right) \tag{4}$$

Making use of this expression, we transform Eq.(3) into

$$\frac{1}{f^2}\frac{dT_{n+1}}{dt}+\sum_{i=1}^{n}\left(\frac{dX_{1i}}{dx_i}\right)^2+\frac{1}{f^2}F-\frac{r^2}{4}\left[\frac{1}{f^5}\frac{d^2f}{dt^2}-\frac{2}{f^6}\left(\frac{df}{dt}\right)^2\right]=0 \tag{5}$$

The variables in this equation are separable if

$$F=f^2\left[\psi_1(x_1)+\psi_2(x_2)+\ldots+\psi_n(x_n)+\eta(t)\right]+$$
$$+\frac{r^2}{4}\left[\frac{1}{f^3}\frac{d^2f}{dt^2}-\frac{2}{f^4}\left(\frac{df}{dt}\right)^2\right] \tag{6}$$

Here $\psi_1(x_1), \ldots, \psi_n(x_n)$, $\eta(t)$ are arbitrary functions. The total integral of Eq. (1) in the case where the function F satisfies condition (6) is given by

$$S = -\frac{r^2}{4f^3}\frac{df}{dt} - \int (C_1 + \eta(t)) f^2 \, dt \pm \int \sqrt{C_1 - C_2 - \psi_1(x_1)} \, dx_1 \pm$$
$$\pm \int \sqrt{C_2 - C_3 - \psi_2(x_2)} \, dx_2 \pm \ldots \pm \int \sqrt{C_{n-1} - C_n - \psi_{n-1}(x_{n-1})} \, dx_{n-1} \pm$$
$$\pm \int \sqrt{C_n - \psi_n(x_n)} \, dx_n + C_{n+1}$$

where $C_1, C_2, \ldots, C_{n+1}$ are arbitrary constants. As is evident from the latter expression, the solution of Hamilton-Jacobi equation (1) obtained by the proposed method generally cannot be obtained by separating variables.

BIBLIOGRAPHY

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